Understanding Euler's Constant e

Bopa Rai

1. The Classroom and the Rabbits

"Alright, settle down, you lot, settle down!" Bopa Rai's voice, a gravelly rumble that could probably crack concrete, echoed through the classroom. The young cadets, still fidgeting in their seats, gradually quieted, their attention drawn to the old man at the front. His uniform, though crisp, bore the subtle marks of countless years of service, and his eyes, though framed by a network of wrinkles, held a sharp, intelligent glint.

"Today," he began, leaning back against the whiteboard, "we're not talking about drill formations or weapons maintenance. Today, we're talking about something far more fundamental, something that underpins almost everything you'll encounter in the world – from a simple interest calculation to the spread of a virus. We're talking about e, and the beautiful, often maddening, dance of compound growth and decay."

He smiled. "Now, I know what some of you are thinking: 'Bopa Rai, are you going to put us to sleep with some dusty old math equations?' Not today, cadets. Today, I'm going to tell you a story. A story that reminds me of a rather persistent rabbit."

Bopa Rai straightened up. "Imagine, if you will, a vast, emerald-green field. And in this field, there's a magnificent, plump carrot. Mr. Growth, our rabbit, loves carrots. Each hop covers half the remaining distance to the carrot. So: half, then half of what's left, and so on."

"Does he ever reach the carrot?"

No. He gets closer and closer, infinitesimally so, but never arrives. This is compound growth in motion.

"Now meet Ms. Decay," Bopa Rai continued. "She's fleeing a fox and hops half the remaining distance to her burrow each time. Again, she never quite reaches it. She's approaching zero – decay."

"This is the paradoxical beauty of e: self-referential change."

He scrawled a single letter on the board: **e**. "It's everywhere – population growth, cooling tea, flowing electricity, even in randomness."

2. Definitions and Approximations of e

2.1 As a Limit (Bernoulli's Original Insight)

Discovered while exploring compound interest:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \tag{1}$$

Examples:

$$n = 100 \Rightarrow (1 + 1/100)^{100} \approx 2.70481$$
$$n = 1,000 \Rightarrow (1 + 1/1000)^{1000} \approx 2.71692$$
$$n = 1,000,000 \Rightarrow (1 + 1/1{,}000{,}000)^{1{,}000{,}000} \approx 2.718280$$

2.2 As the Base of the Natural Logarithm (Area Under a Curve)

Defined via the integral:

$$\int_1^e \frac{1}{x}\,dx = 1 \tag{2}$$

2.3 Using Continued Fractions

Euler's continued fraction expansion:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}} \tag{3}$$

2.4 As an Infinite Series (Euler's Factorial Definition)

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{4}$$

This series converges rapidly and is widely used for computing e.